New Exact Travelling Wave Solutions for Nonlinear Equations with the help of Auxiliary Equation Method

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Abstract

In this paper, by using the solutions of an auxiliary ordinary differential equation, a direct algebraic method is described to construct the exact travelling wave solutions for Improved Korteweg-de Vries Equation, Korteweg-de Vries Equation, Modified Equal Width (MEW) Equation and Modified Regularized Equal Width Equation. It is the method which can be adapted to solve nonlinear partial differential equations.

Keywords: travelling wave solutions; Improved Korteweg-de Vries (IKdv) Equation; Korteweg-de Vries (Kdv) Equation; Modified Equal Width (MEW) Equation; Modified Regularized Equal Width (MREW) Equation

1. Introduction

The investigation for the travelling wave solutions of nonlinear partial differential equations plays an important role in the study of nonlinear physical phenomena. Nonlinear phenomena appears in a wide variety of scientific applications such as plasma physics, solid state physics, optical fibers, biology, fluid dynamics and chemical kinetics. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations. Because of the increased interest in the theory of waves, a broad range of analytical methods was used in the analysis of these scientific models. Mathematical modeling of many physical systems leads to nonlinear ordinary or partial differential equations in various fields of physics and engineering. An effective method is required to analyze the mathematical model which provides solutions conforms to physical reality. Common analytic procedures linearize the system or assume that nonlinearities are relatively insignificant. Such assumptions, sometimes strongly, affect the solution with respect to the real physics of the phenomenon. In recent years, new exact solutions may help to find new phenomena. Thus seeking exact solutions of nonlinear ordinary or partial differential equation is great of importance. Various powerful mathematical methods are such as inverse scattering method [1], Backlund transformation method [2, 3], homogeneous balance method [4], tanh [5-7], extended tanh method [8-10], sine-cosine method [11-13], pseudo spectral method [14], Jacobi elliptic method [15-16] and F-expansion method [17-19].

2. The auxiliary equation method

In this section we will explain the auxiliary equation method. Suppose we are given nonlinear partial differential equation \( u(x,t) \) in the form:

\[
Z(u, u_t, u_x, u_{xx}, u_{tt}, u_{xxx}, u_{xxt}, \ldots) = 0
\]  

(1)
\( u_t \) and \( u_x \) etc. are the partial derivatives of \( u \) with respect to \( t \) and \( x \), respectively. We assume that equation (1) admits travelling wave solution. We use the transformation

\[
u(x, t) = g(\chi); \quad \chi = k(x - ct) + \chi_0 \tag{2}\]

where \( \chi_0 \) is a constant, \( k \) and \( c \) are the wave number and speed of the travelling wave respectively.

This enables us to use the following changes:

\[
\frac{\partial u}{\partial t} = -kc \frac{dg}{d\chi}, \quad \frac{\partial u}{\partial x} = k \frac{dg}{d\chi}, \quad \frac{\partial^2 u}{\partial x^2} = k^2 \frac{d^2 g}{d\chi^2}, \quad \frac{\partial^3 u}{\partial x^3} = k^3 \frac{d^3 g}{d\chi^3}, \quad \frac{\partial^3 u}{\partial x^2 \partial t} = -ck^3 \frac{d^3 g}{d\chi^3}, \ldots
\]

Using the above transformation the nonlinear partial differential equation (1) is transformed to nonlinear ordinary differential equation:

\[
H(g, g', g'', g''', \ldots) = 0 \tag{3}
\]

By virtue of the above method we assume that the solution of Eq. (4) is of the form

\[
g(\chi) = \sum_{i=0}^{M} a_i F^i(\chi) \tag{4}
\]

where \( F(\chi) \) is the solution of the auxiliary ordinary differential equation

\[
(F'(\chi))^2 = q_2 F^2(\chi) + q_3 F^3(\chi) + q_4 F^4(\chi) \tag{5}
\]

where \( q_2, q_3, q_4 \) are real parameters, and hence holds for \( F(\chi) \).

\[
\begin{align*}
F'F'' &= q_2 FF' + \frac{3}{2} q_3 F^2 F' + 2q_4 F^3 F' \\
F'' &= q_2 F' + \frac{3}{2} q_3 F^2 F' + 2q_4 F^3 \\
F''' &= q_2 F'' + 3q_3 F F' + 6q_4 F^2 F'
\end{align*} \tag{6}
\]

Integer \( M \) in (4) can be determined by considering homogeneous balance between the nonlinear terms and the highest derivatives of \( g(\chi) \) in Eq. (3).

In this paper, we present new types solitary wave solutions of Eq. (5). We shall seek the explicitly solitary wave solutions of some nonlinear evolution equations by using the following new solitary wave solutions of Eq. (5):

\[
F(\chi) = -\frac{q_2}{\sqrt{4q_4}} \left( 1 \pm \frac{\sinh(\sqrt{q_2} \chi)}{\cosh(\sqrt{q_2} \chi) \pm 1} \right), \quad q_2, q_4 > 0, q_3^2 - 4q_2 q_4 = 0 \tag{7}
\]
\[
F(\chi) = \begin{cases} 
-\frac{q_2 q_3 \text{sech} \left( \frac{\sqrt{q_2}}{2} \chi \right)}{q_2^2 - q_2 q_4 \left(1 - \tanh \left( \frac{\sqrt{q_2}}{2} \chi \right) \right)^2}, & q_2 > 0 \\
\frac{2q_2 \text{sech} \left( \sqrt{q_2} \chi \right)}{q_3^2 - 4q_2 q_4 - q_3 \text{sech} \left( \sqrt{q_2} \chi \right)}, & q_3^2 - 4q_2 q_4 > 0, q_2 > 0
\end{cases}
\]  

And then by using some significant special solutions of the auxiliary ordinary differential equation, some famous equations are investigated and exact solutions are explicitly obtained with the aid of symbolic computation [17].

As a result, we obtain new explicit solitary wave solutions of some nonlinear evolution equations.

3. Exact travelling wave solutions of IKdv equation

The IKdv equation is given by

\[ u_t + uu_x - u_{xxt} + u_{xxx} = 0 \]  

where \( \alpha \) is a real constant. Making the transformation

\[ u(x, t) = g(\chi) ; \quad \chi = k(x - ct) + \chi_0 \]

Eq. (9) becomes

\[ -c g' + gg' + k^2 g''' + ckg''' = 0 \]  

Following the balance act procedure we balance the highest order of derivative \( g''' \) with the highest power nonlinear term \( gg' \), yields \( M = 2 \). Therefore we may choose the solution of Eq. (10) in the form

\[ g(\chi) = a_0 + a_1 F + a_2 F^2 \]  

where \( a_0, a_1 \) are constants to be determined later. It is easy to deduce that

\[ g'(\chi) = a_1 F' + 2a_2 FF' \]  

\[ g''(\chi) = a_1 F'' + 2a_2 (F')^2 + 2a_2 F F'' \]  

\[ g'''(\chi) = a_1 q_2 F + \left(\frac{3}{2} a_1 q_3 + 4 a_2 q_2\right) F^2 + (2a_1 q_4 + 5a_2 q_3) F^3 + 6a_2 q_4 F^4 \]

Substituting (12)-(14) into Eq. (10) and setting each coefficient of \( F^i F'^i (i = 0,1,2,3) \) to zero,
yields a set of equations for $a_i$ ($i = 0,1,2$),

(15) \[-ca_1 + a_0a_1 + cka_1q_2 + k^2a_1q_2 = 0\]

(16) \[a_1^2 - 2ca_2 + 2a_0a_2 + 8cka_2q_2 + 8k^2a_2q_2 + 3cka_4q_3 + 3k^2a_1q_3 = 0\]

(17) \[3a_1a_2 + 15cka_2q_3 + 15k^2a_2q_3 + 6cka_4 + 6k^2a_1q_4 = 0\]

(18) \[2a_2^2 + 24cka_2q_4 + 24k^2a_2q_4 = 0\]

From the solutions of the system, we can be found

\[a_0 = c - ckq_2 - k^2q_2, \quad a_1 = -6(ckq_3 + k^2q_3), \quad a_2 = -12(ckq_4 + k^2q_4), \quad q_2 = \frac{q_3^2}{4q_4}, q_4 \neq 0\] (19)

with the aid of Mathematica. Substituting (19) into (11) with (7) and (8), we obtain new solution of Eq. (9)

**Solution1:**

\[u(x,t) = (c - ckq_2 - k^2q_2) - 6(ckq_3 + k^2q_3) \left[ -\sqrt{\frac{q_2}{4q_4}} \left( \frac{1}{1 + \frac{\sinh(\sqrt{q_2}k(x - ct) + \chi_0))}{\cosh(\sqrt{q_2}k(x - ct) + \chi_0))} \right) \right] - 12(ckq_4 + k^2q_4) \]

where $q_2, q_4 > 0$.

**Solution2:**

\[u(x,t) = (c - ckq_2 - k^2q_2) - 6(ckq_3 + k^2q_3) \left[ -\frac{q_2q_3 sech^2(\sqrt{\frac{q_2}{2}}(k(x - ct) + \chi_0))}{q_3^2 - q_2q_4 \left[ 1 - tanh(\sqrt{\frac{q_2}{2}}(k(x - ct) + \chi_0)) \right]} \right] \]

\[ - 12ck^2q_4 \left[ -\frac{q_2q_3 sech^2(\sqrt{\frac{q_2}{2}}(k(x - ct) + \chi_0))}{q_3^2 - q_2q_4 \left[ 1 - tanh(\sqrt{\frac{q_2}{2}}(k(x - ct) + \chi_0)) \right]} \right]^2 \]

where $q_2 > 0 \& q_3^2 - 4q_2q_4 > 0$.

**Solution3:**
\[ u(x, t) = (c - ckq_2 - k^2q_2) \]
\[ -6(ckq_3 + k^2q_3) \left[ \frac{2 q_2 \text{sech} \left( \sqrt{q_2} (k(x - ct) + x_0) \right)}{\sqrt{q_2^2 - 4q_2q_4 - q_3 \text{sech}(\sqrt{q_2}(k(x - ct) + x_0))} \right] \]
\[ -12ck^2q_4 \left[ \frac{2 q_2 \text{sech} \left( \sqrt{q_2} (k(x - ct) + x_0) \right)}{\sqrt{q_2^2 - 4q_2q_4 - q_3 \text{sech}(\sqrt{q_2}(k(x - ct) + x_0))} \right]^2 \]

where \( q_2 > 0 \) & \( q_3^2 - 4q_2q_4 > 0 \).

### 4. Exact travelling wave solutions of Kdv equation

The Kdv equation is given by

\[ u_t + auu_x + u_{xxx} = 0 \]  

(20)

Making the transformation

\[ u(x, t) = g(\chi); \quad \chi = k(x - ct) + x_0 \]

Eq. (20) becomes

\[ -c \ g' + \alpha gg' + k^2 g''' = 0 \]  

(21)

Following the balance act procedure we balance the highest order of derivative \( g''' \) with the highest power nonlinear term \( gg' \), yields \( M = 2 \). Therefore we may choose the solution of Eq. (21) in the form

\[ g(\chi) = a_0 + a_1F + a_2F^2 \]  

(22)

where \( a_0, a_1 \) are constants to be determined later. It is easy to deduce that

\[ g'(\chi) = a_1F' + 2a_2FF' \]  

(23)

\[ g''(\chi) = a_1F'' + 2a_2(F')^2 + 2a_2FF'' \]

\[ g'''(\chi) = a_1(q_2F + \frac{3}{2}q_3F^2 + 2q_4F^3) + 2a_2(q_2F^2 + q_3F^3 + q_4F^4) + 2a_2F(q_2F^2 + q_3F^3 + q_4F^4) \]

Substituting (23)-(26) into Eq. (21) and setting each coefficient of \( F^iF'(i = 0,1,2,3) \) to zero, yields a set of equations for \( a_i \) (i = 0,1,2,3),

\[ -ca_1 + \alpha a_0a_1 + k^2a_1q_2 = 0 \]  

(27)

\[ aa_1^2 - 2ca_2 + 2aa_0a_2 + 8k^2a_2q_2 + 3k^2a_1q_3 = 0 \]  

(28)
\[ 3\alpha a_1 a_2 + 15k^2 a_2 q_3 + 6k^2 a_1 q_4 = 0 \]  
\[ 2\alpha a_2^2 + 24k^2 a_2 q_4 = 0 \]

(29)  
(30)

From the solutions of the system, we can be found

\[ a_0 = \frac{c-k^2 q_2}{\alpha}, \quad a_1 = -\frac{6k^2 q_3}{\alpha}, \quad a_2 = -\frac{12k^2 q_4}{\alpha} \]
\[ 4q_2 q_4 = q_3^2, \quad q_2 > 0 \]

(31)

with the aid of Mathematica. Substituting (31) into (22) with (7) and (8), we obtain new solution of Eq. (20)

**Solution1:**

\[
  u(x, t) = \left( \frac{c - k^2 q_2}{\alpha} \right) - \frac{6k^2 q_3}{\alpha} \left[ -\frac{q_2}{4q_4} \left( 1 \pm \frac{\sinh(\sqrt{q_2}(k(x - ct) + \chi_0))}{\cosh(\sqrt{q_2}(k(x - ct) + \chi_0))} \right) \right] 
  - \frac{12k^2 q_4}{\alpha} \left[ -\frac{q_2}{4q_4} \left( 1 \pm \frac{\sinh(\sqrt{q_2}(k(x - ct) + \chi_0))}{\cosh(\sqrt{q_2}(k(x - ct) + \chi_0))} \right) \right]^2
\]

where \( q_2, q_4 > 0 \).

**Solution2:**

\[
  u(x, t) = \left( \frac{c - k^2 q_2}{\alpha} \right) - \frac{6k^2 q_3}{\alpha} \left[ \frac{-q_2 q_3 \sech^2(\sqrt{q_2/2}(k(x - ct) + \chi_0))}{q_3^2 - q_2 q_4 \left[ 1 - \tanh(\sqrt{q_2/2}(k(x - ct) + \chi_0)) \right]} \right] 
  - \frac{12k^2 q_4}{\alpha} \left[ \frac{-q_2 q_3 \sech^2(\sqrt{q_2/2}(k(x - ct) + \chi_0))}{q_3^2 - q_2 q_4 \left[ 1 - \tanh(\sqrt{q_2/2}(k(x - ct) + \chi_0)) \right]} \right]^2
\]

where \( q_2 > 0 \) & \( q_3^2 - 4q_2 q_4 > 0 \).

**Solution3:**

\[
  u(x, t) = \left( \frac{c - k^2 q_2}{\alpha} \right) - \frac{6k^2 q_3}{\alpha} \left[ \frac{2q_2 \sech(\sqrt{q_2}(k(x - ct) + \chi_0))}{\sqrt{q_3^2 - 4q_2 q_4 - q_3 \sech(\sqrt{q_2}(k(x - ct) + \chi_0))}} \right] 
  - \frac{12k^2 q_4}{\alpha} \left[ \frac{2q_2 \sech(\sqrt{q_2}(k(x - ct) + \chi_0))}{\sqrt{q_3^2 - 4q_2 q_4 - q_3 \sech(\sqrt{q_2}(k(x - ct) + \chi_0))}} \right]^2
\]

where \( q_2 > 0 \) & \( q_3^2 - 4q_2 q_4 > 0 \).
5. Exact travelling wave solutions of MEW equation

The MEW equation is given by

\[ u_t + u^2 u_x - u_{xxt} = 0 \]  

(32)

Making the transformation

\[ u(x, t) = g(\chi) \; ; \; \chi = k(x - ct) + \chi_0 \]

Eq. (32) becomes

\[-cg' + g^2 g' + k^2c g''' = 0 \]

(33)

Following the balance act procedure we balance the highest order of derivative \(g''\) with the highest power nonlinear term \(g^3\), yields \(M = 1\). Therefore we may choose the solution of Eq. (33) in the form

\[ g(\chi) = a_0 + a_1 F \]

(34)

where \(a_0, a_1\) are constants to be determined later. It is easy to deduce that

\[ g'(\chi) = a_1 F' \]

(35)

\[ g''(\chi) = a_1 F'' = a_1 \left( q_2 F + \frac{3}{2} q_3 F^2 + 2q_4 F^3 \right) \]

\[ = a_1 q_2 F + \frac{3}{2} a_1 q_3 F^2 + 2a_1 q_4 F^3 \]

(36)

Substituting (34) & (36) into Eq. (33) and setting each coefficient of \(F^i (i = 0,1,2)\) to zero, yields a set of equations for \(a_i \) \((i = 0,1)\),

\[ -ca_1 + a_0^2 a_1 + ck^2 a_1 q_2 = 0 \]

(37)

\[ 2a_0 a_1^2 + 3ck^2 a_1 q_3 = 0 \]

(38)

\[ a_1^3 + 6ck^2 a_1 q_4 = 0 \]

(39)

From the solutions of the system, we can be found

\[ a_0 = -\sqrt{c - ck^2 q_2} \; \; , \; \; a_1 = -\frac{3ck^2 q_3}{2a_0} \; , \; \; 3q_3^2 - 8q_2 q_4 \neq 0 \; \; , \; \; k = \pm \frac{2\sqrt{2}\sqrt{q_4}}{\sqrt{-3q_3^2 + 8q_2 q_4}} \]

(40)

\[ a_0 = \sqrt{c - ck^2 q_2} \; \; , \; \; a_1 = -\frac{3ck^2 q_3}{2a_0} \; , \; \; 3q_3^2 - 8q_2 q_4 \neq 0 \; \; , \; \; k = \pm \frac{2\sqrt{2}\sqrt{q_4}}{\sqrt{-3q_3^2 + 8q_2 q_4}} \]

(41)

with the aid of Mathematica. Substituting (40) & (41) into (34) with (7) and (8), we obtain new solution of Eq. (32)

**Solution1:**

\[ u(x, t) = -\sqrt{c - ck^2 q_2} - \left( \frac{3ck^2 q_3}{2a_0} \right) \left[ \frac{q_2}{4q_4} \left( 1 \pm \frac{\sinh(\sqrt{q_2}(k(x - ct) + \chi_0))}{\cosh(\sqrt{q_2}(k(x - ct) + \chi_0))} \right) \right] \]
And

\[ u(x, t) = \sqrt{c - ck^2} q_2 - \left( \frac{3ck^2q_3}{2a_0} \right) \left[ -\frac{q_2}{4q_4} \left( 1 \pm \frac{\sinh(\sqrt{q_2}(k(x-ct) + \chi_0))}{\cosh(\sqrt{q_2}(k(x-ct) + \chi_0))} \right) \right] \]

where \( q_2, q_4 > 0 \).

Solution2:

\[ u(x, t) = -\sqrt{c - ck^2} q_2 - \left( \frac{3ck^2q_3}{2a_0} \right) \left[ -\frac{q_2q_3}{q_3 - q_2q_4} \left( 1 - \tanh(\frac{\sqrt{q_2}}{2}(k(x-ct) + \chi_0)) \right) \right] \]

And

\[ u(x, t) = \sqrt{c - ck^2} q_2 - \left( \frac{3ck^2q_3}{2a_0} \right) \left[ -\frac{q_2q_3}{q_3 - q_2q_4} \left( 1 - \tanh(\frac{\sqrt{q_2}}{2}(k(x-ct) + \chi_0)) \right) \right] \]

where \( q_2 > 0 \).

Solution3:

\[ u(x, t) = -\sqrt{c - ck^2} q_2 - \left( \frac{3ck^2q_3}{2a_0} \right) \left[ \frac{2q_2 \text{ sech} \left( \frac{\sqrt{q_2}}{2}(k(x-ct) + \chi_0) \right)}{\sqrt{q_3^2 - 4q_2q_4 - q_3 \text{ sech}(\sqrt{q_2}(k(x-ct) + \chi_0))}} \right] \]

And

\[ u(x, t) = \sqrt{c - ck^2} q_2 - \left( \frac{3ck^2q_3}{2a_0} \right) \left[ \frac{2q_2 \text{ sech} \left( \frac{\sqrt{q_2}}{2}(k(x-ct) + \chi_0) \right)}{\sqrt{q_3^2 - 4q_2q_4 - q_3 \text{ sech}(\sqrt{q_2}(k(x-ct) + \chi_0))}} \right] \]

where \( q_2 > 0 \) & \( q_3^2 - 4q_2q_4 > 0 \).

Results and Discussion

The auxiliary equation method was successfully used to establish solitary wave solutions. Many well known nonlinear wave equations were handled by this method to show the new solutions. The performance of the auxiliary equation method is reliable and effective and gives more solutions. The applied method will be used in further works to establish more entirely new solutions for other kinds of nonlinear equations. The availability of computer systems like Mathematica or Maple facilitates the tedious algebraic calculations. The method which we have proposed in this paper is also a standard, direct and computerizes able method, which allows us to do complicated and tedious algebraic calculations.
References


